Synchronization of the Lorenz system through continuous feedback control

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(Received 22 January 1996)

We show that the Lorenz system can be synchronized through the continuous feedback control method and study how the synchronization efficiency is related to the choice of the perturbation applied to the system. [S1063-651X(96)04106-2]

PACS number(s): 05.45.+b

Since it was discussed by Fujisaka and Yamada [1] and demonstrated by Pecora and Carroll [2], synchronization between two chaotic systems has received considerable attention. The most diffused approach to the problem is based on feedback control, which has been clearly described in the pioneering work of Ott, Grebogi, and York (OGY) [3]. Though the application of the OGY method requires a permanent analysis of the state of the system, it deals with discrete temporal changes of some parameter, which makes the method efficiency sensible to the presence of noise. To overcome such limitations Pyragas [4] proposed an alternative approach to chaos control based on a small time continuous perturbation. Synchronization of two chaotic systems by continuous feedback control was later studied by Kapitaniak [5] and Malescio [6]. This method is applied here to achieve synchronization of the Lorenz system. We focus on an important point outlined in our previous work [6], i.e., the dependence of the synchronization efficiency on the perturbation used as a feedback. We will show how the choice of the output variable, as well as that of the equation perturbed, may influence sensibly the synchronizing effect of the perturbation.

Given two chaotic systems

$$\dot{x}=f(x),\qquad \dot{y}=f(y), \tag{1}$$

with $x,y\in\mathbb{R}^n$, called A and B, respectively, let us assume that some state variable, i.e., x_j and y_j $(j=1,\dots,n)$, can be measured. The quantity

$$F_j(t)=K[y_j(t)-x_j(t)], \tag{2}$$

where K is an adjustable positive weight, can then be used as a negative feedback introduced into system A to force its solution over that of system B, so that synchronization eventually follows. Since in this regime $x_j(t)=y_j(t)$, $F_j(t)$ becomes zero and the two systems are practically uncoupled, thus obeying the same dynamics as in absence of the perturbation.

We apply the above outlined procedure to the Lorenz system [7]:

$$\dot{x}_i=f_i(x_1,x_2,x_3), \tag{3}$$

where $i=1,2,3$, and $f_1=Px_1-Px_2$, $f_2=-x_1x_3+Rx_1-x_2$, $f_3=x_1x_2-Bx_3$. The parameters P, R, and B are chosen so to correspond to a chaotic behavior $(P=10,\ R=28,\ B=8/3)$. In order to achieve synchronization, the two systems are coupled by perturbing one of them (called A in the following) through the addition of a feedback control. Different ways of perturbing system A are possible, depending on the choice of the output variable and on that of the equation perturbed (we will assume for simplicity that only one equation at a time can be perturbed). The global system obtained by adding the perturbation $F_j(t)$ to the equation governing the evolution of $x_k(t)$, can be written

$$\begin{aligned}\dot{x}_k&=f_k(x_1,x_2,x_3)+F_j(t),\\ \dot{x}_l&=f_l(x_1,x_2,x_3),\quad l\neq k,\\ \dot{y}_i&=f_i(y_1,y_2,y_3),\end{aligned} \tag{4}$$

where $i,l=1,2,3$. We considered all the possible combinations of k and j (with $k,j=1,2,3$) and in the following we will refer to each of these different ways of perturbing the system with the label kj, where k indicates the equation modified and j the output variable. We exploited a numerical solution of system (4) using the fourth order Runge-Kutta method. Calculations were carried out in double precision [8] with time step $dt=0.01$ (test runs were also performed with a time step of 0.001). As a synchronization criterion we require that the distance $D(t)$ in the phase space between the orbits of the two coupled systems is of the order of the precision of the computer used, i.e., $D(t)<\epsilon$, where $D(t)=\{[x_1(t)-y_1(t)]^2+[x_2(t)-y_2(t)]^2+[x_3(t)-y_3(t)]^2\}^{1/2}$ and $\epsilon=10^{-14}$. For any given perturbation the results are averaged over $N=100$ independent runs with randomly chosen initial conditions. Thus we can define the synchronization fraction $\lambda=N_s/N$, where N_s is the number of runs for which synchronization is attained, with the required precision, within the maximum run length allowed $(5\times10^5$ iterations).

Figures 1(a)–1(c) show how λ depends on the coupling stiffness K when the equation governing, respectively, the evolution of $x_1(t)$, $x_2(t)$, and $x_3(t)$ is modified through the addition of $F_j(t)$, with $j=1,2,3$. According to the different ways of perturbing the system, the synchronization efficiency varies remarkably. In the presence of perturbations 11,12,21,22 we observe a well defined synchronization threshold: when the coupling strength K is greater than a given value, synchronization is attained for all

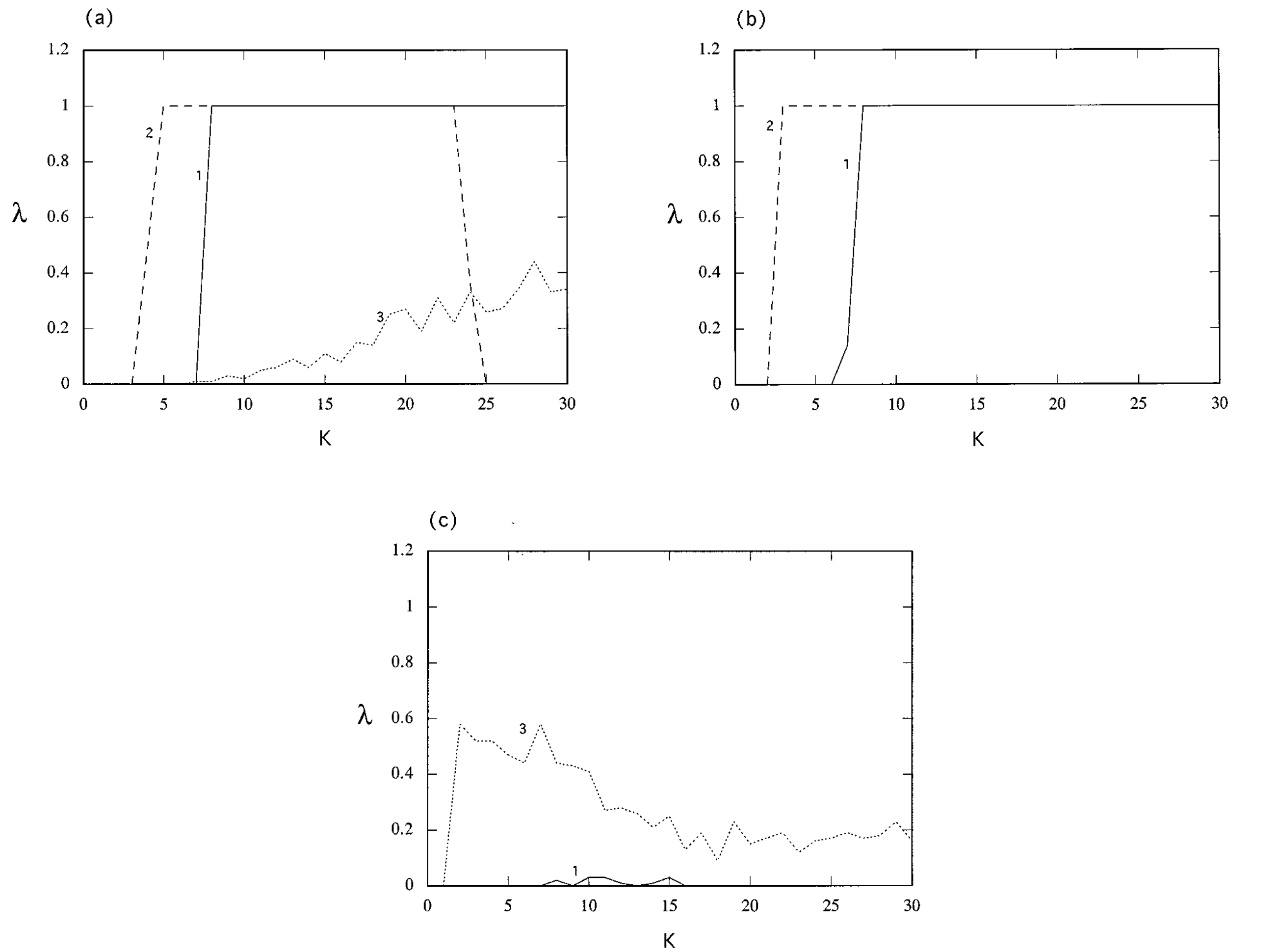

FIG. 1. Plot of the synchronization fraction λ versus coupling stiffness K. The equation governing the evolution of (a) $x_1(t)$, (b) $x_2(t)$, and (c) $x_3(t)$ is perturbed through the addition of $F_j(t)$, with $j=1,2,3$. Curves are labeled with the corresponding value of j and are furthermore differentiated through the line style. Results were averaged over 100 independent runs with run length of 5×10^5 iterations.

the runs performed [in one case (12), however, at higher K's the perturbation loses completely its capacity to induce synchronization]. For perturbations 13,31,33, the synchronization fraction is definitely smaller than one for every value of K considered. Finally we note that perturbations 23 and 32 do not have, in the interval considered, any synchronizing effect at all. This conclusion was verified by performing longer runs (up to 10^6 iterations) [9].

A qualitative understanding of the intriguing variety of behaviors above illustrated may be reached on the basis of simple considerations. We first observe that, once one has selected the equation to be perturbed, the maximum synchronizing effect (evaluated taking into account both the values of λ and the range of K's over which λ is different from zero) is obtained by choosing as output variable the same variable whose time evolution is governed by the equation perturbed [i.e., when, in system (4), $j=k$]. In this case, in fact, the feedback is a direct one and presumably quite efficient in inducing synchronization: the perturbation, proportional to the difference $y_j(t)-x_j(t)$, directly modifies the growth rate of $x_j(t)$, increasing, or decreasing it, according that $x_j(t)$ is smaller, or greater, than $y_j(t)$. For $j\neq k$ the synchronizing effect of the perturbation may be expected to be proportional to the ''degree'' of coupling of the variables $x_j(t)$ and $x_k(t)$. This can be estimated by referring to the physical model schematized by the Lorenz system: a viscous, thermally conducting fluid in a two dimensional rectangular flow region, heated uniformly along the bottom edge in such a way that the temperature difference between fluid at the top and bottom edge is kept constant. In Lorenz approximate model the variable $x_1(t)$ is related to the fluid velocity of a convective circulation in a single eddy that fills the rectangle, $x_2(t)$ describes a temperature distribution with fluid warmer on one side of the rectangle and $x_3(t)$ is related to the vertical profile of temperature [10]. From this picture one expects that $x_1(t)$ is more strongly connected to $x_2(t)$ than to $x_3(t)$, whereas $x_2(t)$ and $x_3(t)$ appear poorly correlated with each other. This scenery is in reasonable agreement with the results shown in Fig. 1.

In conclusion, it appears evident that the choice of the perturbation is a crucial point in the continuous feedback control of chaos. In fact, it affects greatly the synchronization efficiency and may even prove completely ineffective in inducing synchronization. Though the results reported are specific of a given system, they point out the need for a general theory able to predict the optimal choice of the perturbation, i.e., that yielding the maximal efficiency at smaller coupling stiffness. This, in fact, in addition to provide a better understanding of the synchronization process, could be essential in possible practical applications.

This work was supported by the Ministero dell'Università e della Ricerca Scientifica e Tecnologica and by the Consorzio Interuniversitario Nazionale per la Fisica della Materia.

[1] H. Fujisaka and T. Yamada, Prog. Theor. Phys. **69**, 32 (1983).
[2] L.M. Pecora and T.L. Carrol, Phys. Rev. Lett. **64**, 821 (1990).
[3] E. Ott, C. Grebogi, and Y.A. Yorke, Phys. Rev. Lett. **64**, 1196 (1990).
[4] K. Pyragas, Phys. Lett. A **170**, 421 (1992).
[5] T. Kapitaniak, Phys. Rev. E **50**, 1642 (1994).
[6] G. Malescio, Phys. Rev. E **53**, 2949 (1996).
[7] E.N. Lorenz, J. Atmos. Sci. **20**, 130 (1963).
[8] On the RISC processor based computers used this corresponds to a precision of about 10^{-15}.
[9] G. Malescio (unpublished). We extended the investigation also at greater values of the coupling stiffness (up to $K=100$), and observed that λ is generally zero, attaining only occasionally values of the order of 1% or 2%.
[10] J.M.T. Thompson and H.B. Stewart, *Nonlinear Dynamics and Chaos* (Wiley, New York, 1986).